TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 302, Number 2, August 1987

GRADED LIE ALGEBRAS OF THE SECOND KIND

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ABSTRACT. The associated Lie algebra of the Cartan connection for an abstract CR-hypersurface admits a gradation of the second kind. In this article, we give two ways to characterize this kind of graded Lie algebras, namely, geometric characterization in terms of symmetric spaces and algebraic characterization in terms of root systems. A complete list of this class of Lie algebras is given.

0. Introduction. In solving the "equivalence problem" for real hypersurfaces in complex manifolds (cf. **[Ca, T1, CM, T2]**), a principal bundle with a Cartan connection is constructed. The homogeneity which E. Cartan, Tanaka and Chern use in the proof of their theorem arises from a natural grading on the Lie algebra $\mathfrak{G} = \mathfrak{su}(n+1,1)$. That is, $\mathfrak{G}$ can be decomposed as the direct sum

$$\mathfrak{G} = \mathfrak{G}_{-2} + \mathfrak{G}_{-1} + \mathfrak{G}_0 + \mathfrak{G}_1 + \mathfrak{G}_2$$

of vector subspaces $\mathfrak{G}_\alpha$ with $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \subset \mathfrak{G}_{\alpha+\beta}$, $\mathfrak{G}_{-1} \neq \{0\}$, and $\dim \mathfrak{G}_{-2} = 1$.

In this paper, we study this kind of graded Lie algebra. Given an arbitrary $\mathfrak{G}$ graded as above, we construct a quaternionic symmetric space $G^u/SU(2)\cdot H_0^u$ with an antiquaternionic involutive isometry σ. Here G^u is the adjoint group whose Lie algebra is the compact dual of $\mathfrak{G}$, and H_0^u is a certain compact connected subgroup of G^u. The correspondence is one-to-one under suitable equivalence relations (cf. §3).

Let G^c be the adjoint group of $\mathfrak{G}^c$ where $\mathfrak{G}^c$ denotes the complexification of $\mathfrak{G}$. H^c denotes the analytic subgroup of G^c with Lie algebra $(\mathfrak{G}_0 + \mathfrak{G}_1 + \mathfrak{G}_2)^c$, and G denotes the adjoint group with Lie algebra $\mathfrak{G}$. Take $H = G \cap H^c$. (Note that G is isomorphic to the analytic subgroup of G^c with the same Lie algebra $\mathfrak{G}$.) Let K be a maximal compact subgroup of G, and take $K_0 = K \cap H$. It follows that $K/K_0 = G/H$. The fixed point set of σ on $G^u/SU(2)\cdot H_0^u$ contains a connected component which is finitely covered by a Hermitian symmetric space $K/U(1)\cdot K_0^0$ where K_0^0 denotes the identity component of K_0. The associated twistor space or complex contact manifold of $G^u/SU(2)\cdot H_0^u$ is the complex flag manifold $G^c/H^c = G^u/U'(1)\cdot H_0^u$. Moreover, G/H, $K/U(1)\cdot K_0$ and G^c/H^c are related to $G^u/SU(2)\cdot H_0^u$ in the following commutative diagram (cf. Theorem 1 in §2).

$$\begin{array}{ccc} K/K_0 = G/H & \to & G^c/H^c = G^u/U'(1)\cdot H_0^u \\ S^1\downarrow & & S^2\downarrow \\ K/U(1)\cdot K_0 & \to & G^u/SU(2)\cdot H_0^u \end{array}$$

Received by the editors August 1, 1983 and, in revised form, October 21, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17B70, 53C35; Secondary 32F25.

In §4, we give an algebraic characterization of our Lie algebras. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be a Cartan decomposition of $\mathfrak{G}$. Let $\mathfrak{A}_{\mathfrak{p}}$ denote a maximal abelian subspace of $\mathfrak{P}$. The set of roots which do not vanish identically on $\mathfrak{A}_{\mathfrak{p}}$ will be denoted by $\Delta_{\mathfrak{p}}$. Then a simple Lie algebra $\mathfrak{G}$ admits a gradation of the second kind if and only if $\Delta_{\mathfrak{p}}$ contains a root α such that α is a long root in the set of all roots and the multiplicity of the associated restricted root is equal to 1. By the above result and the uniqueness of the gradation, we can give a complete list of our Lie algebras. Finally, we explain a method to read off certain important associated Lie algebras from the so-called extended Satake diagrams of our Lie algebras (cf. Proposition 4.4).

This paper is based on the author's doctoral thesis. In this connection, thanks are due to Professor Tadashi Nagano for his expert guidance. He was generous with his time, ideas, and experience.

1. Preliminaries on graded Lie algebras. In this section we summarize some important facts about graded Lie algebras and deduce what we need for simple graded Lie algebras of the second kind.

A. Let $\mathfrak{G}$ be a Lie algebra over a field F. The additive group of integers will be denoted by $\mathbf{Z}$. Let $(\mathfrak{G}_p)_{p\in\mathbf{Z}}$ be a family of subspaces of $\mathfrak{G}$ such that

(1) $\mathfrak{G} = \sum_{p\in\mathbf{Z}} \mathfrak{G}_p$ (direct sum),

(2) $\dim \mathfrak{G}_p < \infty$ for any $p \in \mathbf{Z}$,

(3) $[\mathfrak{G}_p, \mathfrak{G}_q] \subset \mathfrak{G}_{p+q}$ for any $p, q \in \mathbf{Z}$.

Then we say that the pair $\{\mathfrak{G}, (\mathfrak{G}_p)\}$ or the direct sum $\mathfrak{G} = \sum_{p\in\mathbf{Z}} \mathfrak{G}_p$ is a graded Lie algebra. Let $L = \{\mathfrak{G}, (\mathfrak{G}_p)\}$ and $L' = \{\mathfrak{G}', \mathfrak{G}'_p)\}$ be two graded Lie algebras. Then an isomorphism ϕ of $\mathfrak{G}$ onto $\mathfrak{G}'$ as a Lie algebra is said to be an isomorphism of L onto L' if $\phi(\mathfrak{G}_p) = \mathfrak{G}'_p$ for all p.

B. A semisimple (resp. simple) graded Lie algebra is a graded Lie algebra $L = \{\mathfrak{G}, (\mathfrak{G}_p)\}$ over a field F of characteristic 0 such that $\mathfrak{G}$ is finite dimensional and semisimple (resp. simple). Let $L = \{\mathfrak{G}, (\mathfrak{G}_p)\}$ be a semisimple graded Lie algebra. Then we have the following two results (cf. [**KN**]).

LEMMA B1. *There is a unique element E in the center of $\mathfrak{G}_0$ such that $\mathfrak{G}_p = \{X \in \mathfrak{G} \mid (\operatorname{ad} E)X = pX\}$ for all p.*

LEMMA B2. *Let B denote the Killing form of $\mathfrak{G}$.*

(1) *$B(\mathfrak{G}_p, \mathfrak{G}_q) = 0$ if $p + q \neq 0$.*

(2) *$X \in \mathfrak{G}_p$ and $B(X, \mathfrak{G}_{-p}) = \{0\}$ implies $X = 0$.*

C. From now on, we take F to be the real number field $\mathbf{R}$. A simple graded Lie algebra $L = \{(\mathfrak{G}, (\mathfrak{G}_p)\}$ is said to be of the second kind if $\mathfrak{G}_p = \{0\}$ for $p < -2$, $\mathfrak{G}_{-2} \neq \{0\}$ and $\mathfrak{G}_{-1} \neq \{0\}$. Let $L = \{\mathfrak{G}, (\mathfrak{G}_p)\}$ be a simple graded Lie algebra of the second kind. Then the following holds (cf. [**T2**]).

LEMMA C1. $[\mathfrak{G}_{-1}, \mathfrak{G}_{-1}] = \mathfrak{G}_{-2}$, $[\mathfrak{G}_{-1}, \mathfrak{G}_1] = \mathfrak{G}_0$, $[\mathfrak{G}_1, \mathfrak{G}_1] = \mathfrak{G}_2$, $[\mathfrak{G}_{-2}, \mathfrak{G}_1] = \mathfrak{G}_{-1}$, $[\mathfrak{G}_2, \mathfrak{G}_{-1}] = \mathfrak{G}_1$.

Lemmas B1, B2 and C1 give the next results.

LEMMA C2. *Let* ε *be* 1 *or* -1.
(1) *If* $X \in \mathfrak{G}_0$ *and* $[X, \mathfrak{G}_\varepsilon] = \{0\}$, *then* $X = 0$,
(2) *If* $X \in \mathfrak{G}_\varepsilon$ *and* $[X, \mathfrak{G}_{-\varepsilon}] = \{0\}$, *then* $X = 0$,
(3) *If* $X \in \mathfrak{G}_{2\varepsilon}$ *and* $[X, \mathfrak{G}_{-\varepsilon}] = \{0\}$, *then* $X = 0$.
(4) *If* $X \in \mathfrak{G}_\varepsilon$ *and* $[X, \mathfrak{G}_\varepsilon] = \{0\}$, *then* $X = 0$,
(5) *If* $X \in \mathfrak{G}_\varepsilon$ *and* $[X, \mathfrak{G}_{-2\varepsilon}] = \{0\}$, *then* $X = 0$,
(6) *If* $X \in \mathfrak{G}_{2\varepsilon}$ *and* $[X, \mathfrak{G}_{-2\varepsilon}] = \{0\}$, *then* $X = 0$.

LEMMA C3. *There is an involutive automorphism* σ *of* $\mathfrak{G}$ *having the following properties*:
(1) $\sigma\mathfrak{G}_p = \mathfrak{G}_{-p}$ *for all* p.
(2) *The quadratic form* $B(X, \sigma X)$, $X \in \mathfrak{G}$, *is negative definite.*

D. Henceforth, $L = \{\mathfrak{G}, (\mathfrak{G}_p)\}$ will denote a simple graded Lie algebra of the second kind with $\dim \mathfrak{G}_{-2} = 1$. Let $\mathfrak{B}_0 = \{X \in \mathfrak{G}_0 | [X, \mathfrak{G}_{-2}] = 0\}$.

LEMMA D1. (1) $\mathfrak{G}_0 = \mathbf{R} \cdot E + \mathfrak{B}_0$ (*direct sum*),
(2) $B(\mathfrak{B}_0, E) = \{0\}$.

If we choose a base e^{-2} of $\mathfrak{G}_{-2}$ and let e^2 be the base of $\mathfrak{G}_2$ with $B(e^{-2}, e^2) = 1$, then we have the following result.

LEMMA D2. *Let* $E_2 = [e^2, e^{-2}]$ *and* $\lambda = 1/2(n+1)$ *where* $n = (\dim \mathfrak{G}_{-1})/2 + 1$. *Then* $E_2 = \lambda E$.

For the proof of Lemmas D1 and D2, see [**T2**].

Let $\mathfrak{sl}(2, \mathbf{R})$ be the Lie algebra of the special linear group SL(2, **R**). Let σ be the involutive automorphism of $\mathfrak{G}$ in Lemma C3. Set $A = (2(n+1))^{1/2} e^2$, and $B = (2(n+1))^{1/2} e^{-2}$. Then Lemma D2 implies $[A, B] = E$, $[E, A] = 2A$ and $[E, B] = -2B$, which gives the following.

LEMMA D3. $\mathfrak{G}_{-2} + \mathbf{R}E + \mathfrak{G}_2$ *is isomorphic to* $\mathfrak{sl}(2, \mathbf{R})$.

E. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ be the decomposition of $\mathfrak{G}$ into the eigenspaces of σ for the eigenvalues 1 and -1, respectively. By Lemma C3(2), this is a Cartan decomposition and $\mathfrak{K}$ is a maximal compactly imbedded subalgebra of $\mathfrak{G}$. More precisely, $\mathfrak{K}$ and $\mathfrak{M}$ are equal to $(1+\sigma)(\mathfrak{G}_{-2} + \mathfrak{G}_{-1} + \mathfrak{G}_0)$ and $(1-\sigma)(\mathfrak{G}_{-2} + \mathfrak{G}_{-1} + \mathfrak{G}_0)$ respectively, since $E = [A, \sigma A]$ for some $A \in \mathfrak{G}_{-2}$, $\sigma E = -E$. By Lemma D1 and Lemma C3(1) for $p = 0$, σ leaves $\mathfrak{B}_0$ invariant. (Actually, this follows from our construction of σ.)

Therefore $(1-\sigma)\mathfrak{G}_0$ is equal to $\mathbf{R}E + \mathfrak{B}_0^-$ where $\mathfrak{B}_0 = \mathfrak{K}_0 + \mathfrak{B}_0^-$ is the decomposition of $\mathfrak{B}_0$ into the eigenspaces of σ for the eigenvalues 1 and -1 respectively and $\mathfrak{K}_0$ is obviously equal to $(1+\sigma)\mathfrak{G}_0$. Let $\mathfrak{G}^u = \mathfrak{K} + i\mathfrak{M}$ be the compact real form of $\mathfrak{G}^c$, the complexification of $\mathfrak{G}$.

LEMMA E1. $\mathfrak{G}$ *is not complex*, *i.e.* $\mathfrak{G}$ *does not admit any complex structure by which it becomes a complex Lie algebra.*

PROOF. If $\mathfrak{G}$ is complex, there exists a complex structure J on $\mathfrak{G}$ such that $\operatorname{ad} X \circ J = J \circ \operatorname{ad} X$ for all $X \in \mathfrak{G}$. Take $X = E$. Then an easy computation shows that J leaves $\mathfrak{G}_p$ invariant. In particular, J leaves $\mathfrak{G}_{-2}$ invariant. But the dimension of $\mathfrak{G}_{-2}$ is odd, a contradiction. Q.E.D.

The next result follows from Lemma E1 and the fact that if $\mathfrak{G}^c$ is not simple, then $\mathfrak{G}$ is complex (cf. [**H**]).

LEMMA E2. *Both $\mathfrak{G}^c$ and $\mathfrak{G}^u$ are simple.*

2. Construction of quaternionic symmetric spaces. In this section, we construct so-called quaternionic symmetric spaces. As in §1, let G^c be the adjoint group of $\mathfrak{G}^c$. Let G^u be the analytic subgroup of G^c whose Lie algebra is $\mathfrak{G}^u = \mathfrak{K} + i\mathfrak{M}$. Then G^u is isomorphic to $\operatorname{Int}(\mathfrak{G}^u)$, the adjoint group of $\mathfrak{G}^u$ (cf. [**H**]). Since the Killing form of $\mathfrak{G}^u$ is strictly negative definite, G^u is compact. If G is the analytic subgroup of G^c with Lie algebra $\mathfrak{G}$, then G is isomorphic to $\operatorname{Int}(\mathfrak{G})$.

Let $\mathfrak{G}_{(p)} = \sum_{l \geqslant p} \mathfrak{G}_l$ and set

$$H^c = \left\{ a \in G^c \,|\, \operatorname{Ad}(a) \cdot \mathfrak{G}^c_{(p)} = \mathfrak{G}^c_{(p)} \text{ for any } p \right\}$$

where $\mathfrak{G}^c_{(p)}$ is the complexification of $\mathfrak{G}_{(p)}$. Denote $\mathfrak{G}_{(0)}$ by $\mathfrak{B}$. Then the Lie algebra of H^c is equal to $\mathfrak{B}^c$, the complexification of the Lie algebra $\mathfrak{B}$.

Note that H^c is a closed subgroup of G^c. Actually $G^c = \operatorname{Int} \mathfrak{G}^c = (\operatorname{Aut} \mathfrak{G}^c)^0$ is a closed subgroup of $\mathrm{GL}(\mathfrak{G}^c)$. Therefore H^c is also a closed subgroup of $GL(\mathfrak{G}^c)$.

Now, let $H = G \cap H^c$, so G/H is a submanifold of G^c/H^c. Let K denote the maximal compact subgroup of G whose Lie algebra is $\mathfrak{K}$. Note that $\mathfrak{K} + \mathfrak{B} = \mathfrak{G}$. Since K acts on G/H transitively, $K/K \cap H = G/H$. Similarly, $G^c/H^c = G^u/G^u \cap H^c$. In particular, G^c/H^c is compact. Therefore G^c/H^c is a complex flag manifold, G^c/H^c is simply connected, and $G^u \cap H^c$ is connected. The Lie algebra of $G^u \cap H^c$ is equal to $\mathfrak{K}_0 + i(1 - \sigma)\mathfrak{G}_0$. Recall that $(1 - \sigma)\mathfrak{G}_0 = \mathfrak{B}_0^- + \mathbf{R}E$.

Note that the eigenvalues of E are 0, ± 1 and ± 2 and the ± 1-eigenspaces are nonzero. It follows that *the analytic subgroup of G^u with Lie algebra $i\mathbf{R}E$ is isomorphic to $U(1)$*. Henceforth we denote this subgroup by $U'(1)$. Similarly, *the analytic subgroup of K with Lie algebra $(1 + \sigma)\mathfrak{G}_{-2} = \mathbf{R}\alpha$ with $\alpha = (1 + \sigma)A$ or $A + B$ is isomorphic to $U(1)$*. Henceforth we denote this subgroup by $U(1)$. Let $\mathfrak{B}_0^u$ denote the Lie algebra $\mathfrak{K}_0 + i\mathfrak{B}_0^-$. Then the analytic subgroup H_0^u of G^u with Lie algebra $\mathfrak{B}_0^u$ commutes with $U'(1)$ since E annihilates $\mathfrak{B}_0^u$. The subgroup $U'(1) \cdot H_0^u$ of G^u equals $G^u \cap H^c$ since it is connected. Let U be the analytic subgroup of G^c with Lie algebra $\mathbf{C}\mathfrak{G}_{-2} + \mathbf{C}E + \mathbf{C}\mathfrak{G}_2$. Since $e^{i\pi E}$ is the identity on $\mathfrak{sl}(2, \mathbf{C})$ but is not trivial on $\mathfrak{G}^c$, the homomorphism from U onto $\operatorname{Int}(\mathfrak{sl}(2, \mathbf{C}))$ given by restriction is a nontrivial covering map. Here we identify $\mathbf{C}\mathfrak{G}_{-2} + \mathbf{C}E + \mathbf{C}\mathfrak{G}_2$ with $\mathfrak{sl}(2, \mathbf{C})$ by Lemma D3.

On the other hand, $\operatorname{Ad}_{SL(2,\mathbf{C})}$ is a two to one covering map from $SL(2, \mathbf{C})$ onto $\operatorname{Int}(\mathfrak{sl}(2, \mathbf{C}))$. But $SL(2, \mathbf{C})$ is simply connected. Therefore U must be simply connected too. Since a simply connected Lie group is uniquely determined by its Lie algebra, U is isomorphic to $SL(2, \mathbf{C})$. It follows that the analytic subgroup of G^u or G^c with Lie algebra $\mathfrak{su}(2) = \mathbf{R}\alpha + i\mathbf{R}E + i\mathbf{R}\beta$ (the compact real form of $\mathfrak{sl}(2, \mathbf{C})$)

is isomorphic to $SU(2)$. And $SU(2)$ is a normal subgroup of $SU(2)\cdot H_0^u$ (local product) since $[\mathfrak{su}(2), \mathfrak{B}_0^u] = 0$. We recall that the involutive automorphism σ of $\mathfrak{G}$ can be extended to an automorphism of $\mathfrak{G}^c$. Now induce an automorphism $\tilde{\sigma}$ of $G^u = \mathrm{Int}(\mathfrak{G}^u)$ by $\tilde{\sigma}(g) = \sigma g \sigma^{-1}$. It follows that $\tilde{\sigma}(e^X) = e^{\sigma X}$ for $X \in \mathfrak{G}^u$, so we will use the symbol σ instead of $\tilde{\sigma}$. Remember that K_0 is defined to be $K \cap H$. But

$$K \cap H = K \cap (G \cap H^c) = K \cap H^c = K \cap (G^u \cap H^c)$$
$$= K \cap \big(U'(1) \cdot H_0^u\big) \subset K \cap \big(SU(2) \cdot H_0^u\big).$$

Therefore

$$(*) \qquad U(1) \cdot K_0 = K \cap \big(U(1) \cdot U'(1) \cdot H_0^u\big) \subset K \cap \big(SU(2) \cdot H_0^u\big).$$

PROPOSITION. $K \cap (SU(2) \cdot H_0^u) = U(1) \cdot K_0$.

PROOF. We need only show that $K \cap (SU(2) \cdot H_0^u)$ is contained in $U(1) \cdot K_0$. Let x be an element of $K \cap (SU(2) \cdot H_0^u)$. Then $x = ab$ for some $a \in SU(2)$ and $b \in H_0^u$. Since $K \subset F(\sigma, G^u)$, the set of fixed points of G^u by σ, $ab = x = \sigma x = \sigma a \cdot \sigma b$. So $a^{-1}\sigma a = b \cdot (\sigma b)^{-1} \in SU(2) \cap H_0^u$. (Note that σ leaves $SU(2)$ and H_0^u invariant.) Since every element of H_0^u commutes with elements of $SU(2)$, $SU(2) \cap H_0^u$ is contained in the center of $SU(2)$, denoted by $Z(SU(2))$. But $Z(SU(2)) = \{\pm I_2\}$ where I_2 denotes the 2×2 identity matrix. Here $SU(2)$ is realized in the usual sense and σ is realized as complex conjugation on $SU(2)$. If $a^{-1}\sigma a = I_2$, then $\sigma a = a$ i.e., $a \in F(\sigma, SU(2))$, the set of fixed points of $SU(2)$ by σ. Since $SU(2)$ is compact and simply connected, $F(\sigma, SU(2))$ must be connected. Therefore, $F(\sigma, SU(2)) = U(1)$. In this case, $(*)$ gives the result. If $a^{-1}\sigma a = -I_2$, then $\sigma a = \bar{a}$ and $a\,{}^t\bar{a} = I_2$ imply that $a^{-1} = {}^t\bar{a}$ and ${}^t\bar{a}\bar{a} = -I_2$. So $\bar{a} = -a$ and a is purely imaginary. Therefore, for some $\theta \in \mathbf{R}$,

$$a = \begin{bmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & -i\cos\theta \end{bmatrix}$$

A direct computation shows that

$$a = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix}$$
$$= e^{-\theta\alpha} \cdot e^{i\pi E/2}$$

where $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $E = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ under the canonical isomorphism between $\mathfrak{G}_{-2} + \mathbf{R}E + \mathfrak{G}_2 = \mathbf{R}\alpha + \mathbf{R}E + \mathbf{R}\beta$ and $\mathfrak{sl}(2, \mathbf{R})$. Therefore $a \in U(1) \cdot U'(1)$ and $x \in K \cap (U(1) \cdot U'(1) \cdot H_0^u) = U(1) \cdot K_0$ (see $(*)$). Q.E.D.

Note that K_0 may not be connected and some element of $U(1)$ may not commute with elements of K_0. But any element of the identity component K_0^0 of K_0 commutes with any element of $U(1)$. The proposition above shows that $K/U(1) \cdot K_0$ is a submanifold of $G^u/SU(2) \cdot H_0^u$. We will prove that

(I) $K/U(1) \cdot K_0^0$ is a Hermitian symmetric space of compact type, and

(II) $G^u/SU(2) \cdot H_0^u$ is an irreducible Riemannian symmetric space.

To show (I), define an involutive analytic automorphism $\mathrm{Ad}(s)$ of K by

$$\mathrm{Ad}(s)k = sks^{-1} \quad \text{for } k \in K$$

where $s = \exp(\pi\alpha) \in U(1)$. It follows that $U(1) \cdot K_0^0 = F(\mathrm{Ad}(s), K)^0$, the identity component of the fixed point set of K by $\mathrm{Ad}(s)$, since the Lie algebra of $F(\mathrm{Ad}(s), K)$ is equal to $\mathbf{R}\alpha + \mathfrak{K}_0$. Now, K_0^0 is compact since it is closed in a compact group K. Therefore $\mathrm{Ad}_K(U(1) \cdot K_0^0)$ is compact. We have shown that $(K, U(1) \cdot K_0^0)$ is a Riemannian symmetric pair. Next we are going to show that the symmetric space $K/U(1) \cdot K_0^0$ actually is Hermitian with some complex structure J determined by $U(1)$.

Denote $K/U(1) \cdot K_0^0$ by N and denote the tangent space at $0 = \pi(e)$ by N_0 where $\pi\colon K \to K/U(1) \cdot K_0^0$ is the projection and e is the identity element of K. Identify N_0 with $(1+\sigma)\mathfrak{G}_{-1}$ since $d\mathrm{Ad}(s)_e$ has eigenvalue -1 on $(1+\sigma)\mathfrak{G}_{-1}$ and eigenvalue 1 on $\mathbf{R}\alpha + \mathfrak{K}_0$. Define an endomorphism J_0 of N_0 by $J_0 = \mathrm{Ad}_K(j)$ where $j = \exp(\pi\alpha/2)$. Let Q be any K-invariant Riemannian metric on N. Then

(a) $J_0^2 = -I$ where I is the identity map on N_0,

(b) $Q_0(J_0X, J_0Y) = Q_0(X, Y)$ for $X, Y \in N_0$ since Q_0 is $\mathrm{Ad}_K(U(1) \cdot K_0^0)$-invariant, and

(c) J_0 commutes with each element of the linear isotropy group since $[\alpha, \mathfrak{K}_0] = 0$ and $U(1)$ is contained in the center of $U(1) \cdot K_0^0$.

Therefore (cf. [**H**]) N has a unique K-invariant almost complex structure, denoted by J, such that $J = J_0$ at N_0 and the metric Q is Hermitian with respect to J. Furthermore, J is integrable and with the corresponding complex structure, N is a Hermitian symmetric space. That N is of compact type follows from the fact that the center of $\mathfrak{K}$ is contained in $\mathbf{R}\alpha + \mathfrak{K}_0$. (See Lemma C2.)

To show (II), let $\mathrm{Ad}(s)$ work on G^u. Since the Lie algebra of $F(\mathrm{Ad}(s), G^u)$ is equal to $\mathfrak{su}(2) + \mathfrak{B}_0^u$, it follows that the identity component of $F(\mathrm{Ad}(s), G^u)$ is equal to $SU(2) \cdot H_0^u$. Since $G^u \cap H^c = U'(1) \cdot H^0$ is closed in a compact group G^u, both H_0^u and $\mathrm{Ad}_{G^u}(SU(2) \cdot H_0^u)$ are compact. So $(G^u, SU(2) \cdot H_0^u)$ is a Riemannian symmetric pair. Hence $G^u/SU(2) \cdot H_0^u$ is a Riemannian symmetric space. Next, we show that σ induces an involutive isometry defined on $M = G^u/SU(2) \cdot H_0^u$. Since both $SU(2)$ and H_0^u are left invariant by σ, we can induce an involutive map, denoted by the same symbol σ, on M. Let Q be a G^u-invariant metric on M. We identity M_0, the tangent space at $0 = \pi(e)$, with $\mathfrak{P} = (1+\sigma)\mathfrak{G}_{-1} + i(1-\sigma)\mathfrak{G}_{-1}$ where $\pi\colon G^u \to G^u/SU(2) \cdot H_0^u$ is the projection and e is the identity element of G^u. Denote $SU(2) \cdot H_0^u$ by H' and take $j = \exp(\pi i E/2) \in H'$. For $Z \in (1+\sigma)\mathfrak{G}_{-1}$, $W \in i(1-\sigma)\mathfrak{G}_{-1}$

$$Q_0(Z, W) = Q_0(\mathrm{ad}(j)Z, \mathrm{Ad}(j)W) = Q_0(-W, Z)$$

by the previous computation for $\exp(tiE)$. Therefore $Q_0(Z, W) = 0$. It follows that for $X, Y \in \mathfrak{P} = M_0$,

$$(**) \qquad Q_0(X, Y) = Q_0(\sigma X, \sigma Y) = Q_0(d\sigma_0 X, d\sigma_0 Y).$$

Note that we are using the same symbol σ acting on the group G^u, the Lie algebra $\mathfrak{G}^u$, and the symmetric space M. Now let $p = g \cdot 0$, $g \in G^u$.

$$\begin{aligned} Q_{\sigma \cdot p}(d\sigma_p X, d\sigma_p Y) &= Q_0\big(d(\sigma g)^{-1} \circ d\sigma_p X, d(\sigma g)^{-1} \circ d\sigma_p Y\big) \\ &= Q_0\big(d\sigma_0 \circ dg^{-1} X, d\sigma_0 \circ dg^{-1} Y\big). \end{aligned}$$

Since $d\sigma_p = d(\sigma g) \circ d\sigma_0 \circ dg^{-1}$, the above equation

$$= Q_0(dg^{-1}X, dg^{-1}Y) = Q_p(X, Y)$$

by (**) and since Q is G^u-invariant with the map $x \in M \to g \cdot x$, $g \in G^u$, denoted by g for simplicity. We have shown that σ is an isometry.

Since each element of K is fixed by σ, $N = K/U(1) \cdot K_0$ is the identity component of the fixed point set of σ. Therefore N is a totally geodesic submanifold of M. Note that $(1 + \sigma)\mathfrak{G}_{-1}$ is the Lie triple system of N.

Now,

$$U'(1) \cap H_0^u \subseteq SU(2) \cap H_0^u \subseteq Z(SU(2)) = \{\pm I_2\}$$

and

$$U(1) \cap K_0 = U(1) \cap K \cap (U'(1) \cdot H_0^u)$$
$$= U(1) \cap (U'(1) \cdot H_0^u) = \{\pm I_2\},$$

since $a = bx$ for $a \in U(1)$, $b \in U'(1)$ and $x \in H_0^u$ imply that $b^{-1}a = x \in SU(2) \cap H_0^u \subseteq \{\pm I_2\}$. Therefore a is contained in $U(1) \cap U'(1) = \{\pm I_2\}$. A similar argument shows that $SU(2) \cap (U'(1) \cdot H_0^u)$ is contained in $SU(2) \cap U'(1) = U'(1)$. In summary we have

(A) $U'(1) \cap H_0^u \subseteq SU(2) \cap H_0^u \subseteq \{\pm I_2\}$,

(B) $U(1) \cap K_0 = \{\pm I_2\}$,

(C) $SU(2) \cap (U'(1) \cdot H_0^u) \subseteq U'(1)$.

Now, the canonical map from $U(1)$ onto $U(1) \cdot K_0/K_0$ is two-to-one by (B). Therefore $U(1) \cdot K_0/K_0$ is diffeomorphic to the unit circle S^1 and $SU(2) \cdot H_0^u/U'(1) \cdot H_0^u$ is diffeomorphic to $SU(2)/U'(1)$ by (C). But $SU(2)/U'(1)$ is a simply connected compact 2-dimensional manifold. Therefore it is diffeomorphic to S^2. Finally, M is irreducible by Lemma E2 which says that $\mathfrak{G}^u$ is simple.

Thus we have proved this theorem.

THEOREM 1. *The following diagram commutes.*

$$\begin{array}{ccc} K/K_0 = G/H & \to & G^c/H^c = G^u/U'(1) \cdot H_0^u \\ S^1 \downarrow & & S^2 \downarrow \\ K/U(1) \cdot K_0 & \to & G^u/SU(2) \cdot H_0^u \end{array}$$

Here K/K_0 is fibered by S^1 over $K/U(1) \cdot K_0$ and $G^u/U'(1) \cdot H_0^u$ is fibered by S^2 over $G^u/SU(2) \cdot H_0^u$. The maps $G/H \to G^c/H^c$ and $K/U(1) \cdot K_0 \to G^u/SU(2) \cdot H_0^u$ are imbeddings of submanifolds. Furthermore,

(I) *$K/U(1) \cdot K_0^0$ is a simply connected Hermitian symmetric space of compact type.*

(II) *$G^u/SU(2) \cdot H_0^u$ is a simply connected irreducible Riemannian symmetric space of compact type.*

(III) *$K/U(1)\cdot K_0$ is a connected component of the fixed point set of some involutive isometry σ of $G^u/SU(2)\cdot H_0^u$.*

(IV) *G^u acts effectively on $G^u/SU(2)\cdot H_0^u$.*

Note that G^u is a simple adjoint group which does not admit any normal (continuous or discrete) subgroup. The simple connectivity of $G^u/SU(2)\cdot H_0^u$ follows from the fibration $S^2 \to G^c/H^c \to G^u/SU(2)\cdot H_0^u$ and simple connectivity of the complex flag manifold G^c/H^c. Now we give a definition of a special class of symmetric spaces.

DEFINITION. A quaternionic symmetric space is a Riemannian symmetric space with a quaternionic structure in which the holonomy has quaternion scalar part. (For a description of quaternionic structure, see the Appendix.)

Notice that $G^u/SU(2)\cdot H_0^u$ in Theorem 1 is a compact simply connected quaternionic symmetric space with an antiquaternionic involutive isometry σ and $\sigma(0)=0$. (For the definition of antiquaternionic maps, also see the Appendix.)

Let (M,σ) denote a compact simply connected quaternionic symmetric space M with an antiquaternionic involutive isometry σ. Let (M_1,σ_1) and (M_2,σ_2) be two such pairs with quaternionic structures A_1 and A_2 respectively. Call (M_1,σ_1) and (M_2,σ_2) equivalent if there exists an isometry f from M_1 onto M_2 such that $\sigma_2\circ f = f\circ\sigma_1$ and f carries A_1 to A_2. We show in the Appendix that the last condition can be removed without effect.

Note that equivalence of (M,σ_1) and $(M\sigma_2)$ does not a priori make σ_1 and σ_2 the same.

Let $(G^u/SU(2)\cdot H_0^u,\sigma)$ and $(\tilde G^u/\tilde S\tilde U(2)\cdot \tilde H_0^u,\tilde\sigma)$ be two of them. ($\tilde S\tilde U(2)$ is isomorphic to $SU(2)$.) Since the Cartan decomposition is unique up to conjugacy, we may find an inner automorphism ρ of $\mathfrak{G}$ extended to an automorphism of $\mathfrak{G}^c$ (still denoted by ρ) such that $\rho(\tilde{\mathfrak{K}})=\mathfrak{K}$, $\rho(\tilde{\mathfrak{M}})=\mathfrak{M}$ where $\tilde{\mathfrak{K}}+\tilde{\mathfrak{M}}$ and $\mathfrak{K}+\mathfrak{M}$ are the two Cartan decompositions of $\mathfrak{G}$. Also there is a $\tau\in \mathrm{Ad}\,K$ such that $(\tau\circ\rho)E=E$. Hence $(\tau\circ\rho)\mathfrak{G}_p=\mathfrak{G}_p$. Since $\mathfrak{su}(2)+\mathfrak{B}_0^u=\mathfrak{G}^u\cap(\mathfrak{G}_{-2}+\mathfrak{G}_0+\mathfrak{G}_2)^c$ and $\tilde{\mathfrak{s}}\tilde{\mathfrak{u}}(2)+\tilde{\mathfrak{B}}_0^u=\tilde{\mathfrak{G}}^u\cap(\mathfrak{G}_{-2}+\mathfrak{G}_0+\mathfrak{G}_2)^c$, $\mathrm{Ad}(\tau\circ\rho)$ maps $\tilde S\tilde U(2)\cdot H_0^u$ onto $SU(2)\cdot H_0^u$. Therefore $\mathrm{Ad}(\tau\circ\rho)$ induces an isometry f from $\tilde G^u/\tilde S\tilde U(2)\cdot\tilde H_0^u$ onto $G^u/SU(2)\cdot H_0^u$ and $f\circ\tilde\sigma=\sigma\circ f$. Furthermore, $\mathrm{Ad}(\tau\circ\rho)$ carries $\tilde S\tilde U(2)$ to $SU(2)$ since $\tilde{\mathfrak{s}}\tilde{\mathfrak{u}}(2)=\tilde{\mathfrak{G}}^u\cap(\mathfrak{G}_{-2}^c+\mathbf{C}E+\mathfrak{G}_2^c)$ and $\mathfrak{su}(2)=\mathfrak{G}^u\cap(\mathfrak{G}_{-2}^c+\mathbf{C}E+\mathfrak{G}_2^c)$. Therefore f carries the quaternionic structure of $\tilde G^u/\tilde S\tilde U(2)\cdot\tilde H_0^u$ to that of $G^u/SU(2)\cdot H_0^u$. We have shown that $(\tilde G^u/\tilde S\tilde U(2)\cdot\tilde H_0^u,\tilde\sigma)$ is equivalent to $(G^u/SU(2)\cdot H_0^u,\sigma)$.

3. The reverse process and examples. Let $L=\{\mathfrak{G},(\mathfrak{G}_p)\}$ be a simple graded Lie algebra of the second kind with $\dim\mathfrak{G}_{-2}=1$. By Lemma B1, there is a one-to-one correspondence between the set of L's and the set of the pairs $(\mathfrak{G},E)$ where $E\in\mathfrak{G}$, while $\mathrm{ad}\,E$ is semisimple and has only eigenvalues -2, -1, 0, 1, 2 with the eigenspace of -1 nontrivial and the dimension of the eigenspace of -2 equal to 1. From now on, we use $(\mathfrak{G},E)$ instead of L to specify the role of the gradation and call $(\mathfrak{G},E)$ a gradation of the second kind for $\mathfrak{G}$. Let $(\mathfrak{G},E)$ and $(\tilde{\mathfrak{G}},\tilde E)$ be two gradations of the second kind for $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ respectively. Here $(\mathfrak{G},E)$ and $(\tilde{\mathfrak{G}},\tilde E)$ are equivalent if there is a Lie algebra isomorphism f from $\mathfrak{G}$ onto $\tilde{\mathfrak{G}}$ such that $f(E)=\tilde E$.

Theorem 1 in §2 shows how to map the equivalence class of $(\mathfrak{G}, E)$ to the equivalence class of (M, σ) where M is a compact simply connected quaternionic symmetric space and σ is an antiquaternionic involutive isometry with $\sigma(0) = 0$ for some point $0 \in M$. In this section we show that this map is surjective and injective.

Start with a given compact simply connected quaternionic symmetric space M with an antiquaternionic involutive isometry σ such that $\sigma(0) = 0$ for some point $0 \in M$. Let I_0M denote the identity component of the isometry group of M. The isotropy subgroup of I_0M acting on M at 0 is denoted by H'. Since M is quaternionic and simply connected, $H' = SU(2) \cdot H_0^u$ for some connected compact Lie group H_0^u which centralizes $SU(2)$. Furthermore, $SU(2)$ acts on the complexification $M_0^c = V \otimes W$ of the tangent space at 0, irreducibly on a complex 2-dimensional vector space V, trivially on another complex vector space W. Since σ is antiquaternionic, there is a basis of $\mathfrak{su}(2)$, say

$$\alpha \cong \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad i\beta \cong \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad \text{and} \quad iE \cong \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

such that $\sigma\alpha = \alpha$, $\sigma\beta = -\beta$, $\sigma E = -E$. Here the same symbol σ means $d\,\mathrm{Ad}(\sigma)_e$ where e is the identity element of I_0M. Let $\mathfrak{G}^u$ denote the Lie algebra of $G^u = I_0M$. Then $d\,\mathrm{Ad}(\sigma)_e$ can be extended to an automorphism of the Lie algebra $\mathfrak{G}^c$ where $\mathfrak{G}^c$ is the complexification of $\mathfrak{G}^u$. We will use the same symbol σ for $\mathrm{Ad}(\sigma)$ and $d\,\mathrm{Ad}(\sigma)_e$ whenever no confusion will result. Now $\mathfrak{G}^u$ is decomposed into eigenspaces $\mathfrak{K}$ and $\mathfrak{P}$ of σ for the eigenvalues 1 and -1, respectively. Let $\mathfrak{G} = \mathfrak{K} + i\mathfrak{P}$. We may write $i\mathfrak{P}$ as $\mathfrak{M}$.

LEMMA. *$B_{\mathfrak{G}}(X, \sigma X) < 0$ for $X \in \mathfrak{G}$.*

PROOF. If $X \in \mathfrak{K}$, then $B_{\mathfrak{G}}(X, \sigma X) = B_{\mathfrak{G}}(X, X) = B_{\mathfrak{G}^c}(X, X) = B_{\mathfrak{G}^u}(X, X) < 0$ since $\mathfrak{G}^u$ is compact and simple. If $X \in \mathfrak{M}$, then $B_{\mathfrak{G}}(X, \sigma X) = B_{\mathfrak{G}^c}(X, -X) = B_{\mathfrak{G}^c}(iX, iX) = B_{\mathfrak{G}^u}(iX, iX) < 0$ for the same reason. If $Y \in \mathfrak{M}$, $X \in \mathfrak{K}$, then $B(X, Y) = B(\sigma X, Y) = B(X, \sigma Y) = -B(X, Y)$. Therefore $B(X, Y) = B(Y, X) = 0$. Since $B_{\mathfrak{G}}$ is bilinear, the conclusion follows. Q.E.D.

COROLLARY. *$\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ is a Cartan decomposition.*

Since $\sigma(0) = 0$, $0 \in M$, σ leaves invariant the isotropy group $H' = SU(2) \cdot H_0^u$. It follows that σ leaves H_0^u invariant since it leaves $SU(2)$ invariant and $\mathfrak{su}(2)$ is orthogonal to $\mathfrak{B}_0^u = L(H_0^u)$, the Lie algebra of H_0^u. Therefore $\mathfrak{B}_0^u$ is decomposed into eigenspaces $\mathfrak{K}_0$ and $i\mathfrak{B}_0^-$ of σ for the eigenvalues 1 and -1, respectively where $\mathfrak{B}_0^-$ is contained in $\mathfrak{M}$. Let $\mathfrak{K} = \mathbf{R}\alpha + \mathfrak{K}_0 + \mathfrak{K}_1$ and $\mathfrak{M} = \mathbf{R}\beta + \mathbf{R}E + \mathfrak{B}_0^- + \mathfrak{M}_1$ be the orthogonal decompositions. Let s be the involutive automorphism of $G^u = I_0M$ defined by $s(g) = s_0 g s_0^{-1}$ for $g \in G^u$ where s_0 is the geodesic symmetry at 0. It is easy to see that $\mathfrak{K}_1 + i\mathfrak{M}_1$ is exactly the eigenspace of $(ds)_0$ with eigenvalue -1.

Now $\mathfrak{su}(2)$ acts on $M_0^c = V \otimes W \cong (\mathfrak{K}_1 + \mathfrak{M}_1)^c$ as the adjoint action and irreducibly on V, trivially on W. The action of $\mathfrak{su}(2)$ can be extended to that of $\mathfrak{su}(2)^c = \mathfrak{sl}(2, \mathbf{C})$ naturally so that $\mathfrak{sl}(2, \mathbf{C})$ still acts irreducibly on V and trivially on W. By the representation theory of $\mathfrak{sl}(2, \mathbf{C})$, V has the decomposition $V_{-1} + V_1$ where V_{-1} and V_1 are weight spaces of E for weights -1 and 1. Let $\mathfrak{G}_{-1}^c$, $\mathfrak{G}_1^c$ denote $V_{-1} \otimes W$ and $V_1 \otimes W$ respectively. Then $[E, v_1 \otimes w] = v_1 \otimes w$ and $[E, v_{-1} \otimes w] = -v_{-1} \otimes w$ for $v_1 \otimes w \in \mathfrak{G}_1^c$ and $v_{-1} \otimes w \in \mathfrak{G}_{-1}^c$. Since $(\mathfrak{K}_1 + \mathfrak{M}_1)^c = \mathfrak{G}_{-1}^c + \mathfrak{G}_1^c$ and $\operatorname{ad} E$ leaves $\mathfrak{K}_1 + \mathfrak{M}_1$ invariant, $\mathfrak{K}_1 + \mathfrak{M}_1$ will have the decomposition $\mathfrak{G}_{-1} + \mathfrak{G}_1$ where $\mathfrak{G}_{-1} = \mathfrak{G}_{-1}^c \cap (\mathfrak{K}_1 + \mathfrak{M}_1)$ and $\mathfrak{G}_1 = \mathfrak{G}_1^c \cap (\mathfrak{K}_1 + \mathfrak{M}_1)$. Since $\sigma \mathfrak{G}_{-1} = \mathfrak{G}_1$ and σ leaves $\mathfrak{G}_{-1} + \mathfrak{G}_1$ invariant, it follows that $\mathfrak{K}_1 = (1 + \sigma)\mathfrak{G}_{-1}$, $\mathfrak{M}_1 = (1 - \sigma)\mathfrak{G}_{-1}$.

Let $\mathfrak{G}_{-2} = \mathbf{R}(\alpha + \beta)$, $\mathfrak{G}_2 = \mathbf{R}(\alpha - \beta)$ and $\mathfrak{G}_0 = \mathbf{R}E + \mathfrak{K}_0 + \mathfrak{B}_0^-$. Then it is easy to see that $\mathfrak{G} = \sum_{p \in \mathbf{Z}, p=-2}^2 \mathfrak{G}_p$ is a graded Lie algebra of the second kind with $\dim \mathfrak{G}_{-2} = 1$. Since $\mathfrak{G}^u$ is compact and simple, $\mathfrak{G}^u$ is not complex. Hence $\mathfrak{G}^c$ is simple. Therefore $\mathfrak{G}$ is simple. Now (M, σ) can be obtained through the process in §2 from $(\mathfrak{G}, E)$. Note that $I_0 M$ is centerless as shown in the Appendix. If $(\tilde{M}, \tilde{\sigma})$ is equivalent to (M, σ), the corresponding $\tilde{\mathfrak{G}}$ is isomorphic to $\mathfrak{G}$. Therefore $(\tilde{\mathfrak{G}}, \tilde{E})$ is equivalent to $(\mathfrak{G}, E)$ by the uniqueness of the gradation noted in §4. Thus we have proved the following theorem.

THEOREM 2. *There is a one-to-one correspondence between equivalence classes of graded Lie algebras of the second kind with* $\dim \mathfrak{G}_{-2} = 1$ *and equivalence classes of compact simply connected quaternionic symmetric spaces with antiquaternionic involutive isometries that have fixed points. The correspondence is given by* $(\mathfrak{G}, E) \to (G^u/SU(2) \cdot H_0^u, \sigma)$ *where the quaternionic symmetric space* $G^u/SU(2) \cdot H_0^u$ *with an antiquaternionic involutive isometry* σ *is constructed in* §2.

In the following examples, H^c always contains the center of G^c, denoted by Z^c. It follows that $G \cap Z^c = Z$, $G^u \cap Z^c = Z^u$ and $H \supset Z$ where Z and Z^u denote the centers of G and G^u respectively. Therefore each space in the diagram is isomorphic to the corresponding one in Theorem 1 although G^c is not centerless a priori.

EXAMPLE 1. Take

$$\mathfrak{G} = \mathfrak{sp}(n, \mathbf{R}) = \left\{ \begin{bmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{bmatrix} : \begin{array}{l} X_1,\ X_2,\ X_3 \text{ real } n \times n \text{ matrices} \\ \\ X_2,\ X_3 \text{ symmetric} \end{array} \right\}.$$

Let $G = Sp(n, \mathbf{R})$ be the group of matrices in $GL(2n, \mathbf{R})$ which leave invariant the exterior form $x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n}$ where the action on $\mathbf{R}^{2n} = \{(x_1, x_2, \ldots, x_{2n})\}$ is canonical. This canonical action induces an action on $\mathbf{R}P^{2n-1}$, the real projective space of $\dim 2n - 1$. It is transitive since G contains $U(n)$ (see below). We take H to be the isotropy subgroup at $[1:0:\cdots:0] \in \mathbf{R}P^{2n-1}$. Now

$G^c = Sp(n,C)$, H^c is the isotropy subgroup of G^c acting on $\mathbf{C}P^{2n-1}$ at $[1:0:\cdots:0]$, and $\sigma X = -{}^tX$ for $X \in \mathfrak{G}^c = \mathfrak{sp}(n,\mathbf{C})$. Note that H^c contains $Z^c =$ the center of $Sp(n,\mathbf{C}) = \{\pm I\}$ where I denotes the identity matrix. Let

$$E = \begin{bmatrix} X_1 & 0 \\ 0 & -{}^tX_1 \end{bmatrix}, \qquad X_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}_{n\times n},$$

$$\mathfrak{G}_{+2} = \sigma\mathfrak{G}_{-2} = \left\{ \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix} : X_2 = \begin{bmatrix} s & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}_{n\times n}, s \in \mathbf{R} \right\},$$

$$\mathfrak{G}_{+1} = \sigma\mathfrak{G}_{-1} = \left\{ \begin{bmatrix} X_1 & X_2 \\ 0 & -{}^tX_1 \end{bmatrix} : X_1 = \begin{bmatrix} 0 & x_2 & \cdots & x_n \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}, \right.$$

$$\left. X_2 = \begin{bmatrix} 0 & y_2 & \cdots & y_n \\ y_2 & & & \\ \vdots & & 0 & \\ y_n & & & \end{bmatrix} \right\},$$

$$x_i, y_i \in \mathbf{R}, 2 \leqslant i \leqslant n,$$

$$\mathfrak{G}_0 = \left\{ \begin{bmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{bmatrix} : X_1 = \begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}, \right.$$

$$\left. X_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix} \right\},$$

$$X_2,\ X_3 \text{ symmetric},$$

$$\mathfrak{K} = \left\{ \begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix} : X_1 + {}^tX_1 = 0,\ X_2 \text{ symmetric} \right\} = \mathfrak{u}(n),$$

$$\mathfrak{u}(1) = \left\{ \begin{bmatrix} 0 & X \\ -X & 0 \end{bmatrix} : X = \begin{bmatrix} s & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}_{n\times n}, s \in \mathbf{R} \right\},$$

$$\mathfrak{K}_0 = \left\{ \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & X & & \vdots & A & \\ 0 & & & 0 & & \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & -A & & & X & \\ 0 & & & & & \end{array} \right] : X \text{ skew symmetric, } A \text{ symmetric} \right\}$$

$$= \mathfrak{u}(n-1),$$

$$\mathfrak{B}_0 = \mathfrak{sp}(n-1, \mathbf{R}).$$

Here $H = N_+ \rtimes (\mathbf{R}_+ \times Sp(n-1, \mathbf{R}))$, where $\rtimes$ denotes the semisimple product, N_+ is the analytic subgroup with Lie algebra $\mathfrak{G}_1 + \mathfrak{G}_2$, and $\mathbf{R}_+$ is just the multiplicative group of positive real numbers. Similarly

$$H^c = N_+^c \rtimes \left(\mathbf{C}^* \times Sp(n-1, \mathbf{C})\right)$$

where N_+^c is the analytic subgroup with Lie algebra $(\mathfrak{G}_1 + \mathfrak{G}_2)^c$ and $\mathbf{C}^*$ is the multiplicative group of nonzero complex numbers. Moreover

$$K = U(n), \qquad K_0 = K \cap H = \{\pm I'\} \times U(n-1),$$

$$I' = \left[\begin{array}{cc|cc} 1 & & & \\ & 0 & 0 & \\ \hline & 0 & 1 & \\ & & & 0 \end{array} \right],$$

$$\mathfrak{su}(2) = (1+\sigma)\mathfrak{G}_{-2} + i\left((1-\sigma)\mathfrak{G}_{-2} + \mathbf{R}E\right)$$

$$= \left\{ \left[\begin{array}{cc|cc} ic & & -a+ib & \\ & 0 & & 0 \\ \hline a+ib & & -ic & \\ & 0 & & 0 \end{array} \right] \right\} = \mathfrak{sp}(1,0) = \mathfrak{sp}(1),$$

$$\mathfrak{B}_0^u = \left\{ \left[\begin{array}{cc|cc} 0 & & 0 & \\ & X+iY & & A+iB \\ \hline 0 & & 0 & \\ & -A+iB & & X-iY \end{array} \right] : \begin{array}{l} X \text{ skew symmetric} \\ B, A, Y \text{ symmetric} \end{array} \right\}$$

$$= \mathfrak{sp}(n-1),$$

$$\mathfrak{G}^u = \mathfrak{sp}(n), \qquad G^u = Sp(n).$$

Now we have the following commutative diagram.

$$\begin{array}{l} \mathbf{R}P^{2n-1} = U(n)/\{\pm I'\} \times U(n-1) \\ \qquad \mathbf{R}P^1 = S^1 \downarrow \\ \mathbf{C}P^{n-1} = U(n)/U(1) \times U(n-1) \\ \qquad = Sp(n, \mathbf{R})/H \to Sp(n, \mathbf{C})/H^c = Sp(n)/U(1) \times Sp(n-1) = \mathbf{C}P^{2n-1} \\ \qquad\qquad \mathbf{C}P^1 = S^2 \downarrow \\ \qquad = U(n)/U(1) \cdot \left(\{\pm I'\} \times U(n-1)\right) \to Sp(n)/Sp(1) \times Sp(n-1) = \mathbf{H}P^{n-1} \end{array}$$

EXAMPLE 2. $\mathfrak{G} = \mathfrak{su}(Q) = \mathfrak{su}(p+1, q+1)$ with respect to Q where

$$Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & I_{pq} & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad I_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -1_q \end{bmatrix},$$

1_p denotes the $p \times p$ identity matrix.

$$G = SU(p+q, q+1),$$

$$\sigma X = -{}^t\overline{X} \text{ for } X \in \mathfrak{G},$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0_{n \times n} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad n = p+q,$$

$$\mathfrak{su}(Q) = \left\{ A = \begin{bmatrix} a & {}^t u_1 & {}^t u_2 & it \\ \bar{z}_1 & X_1 & X_2 & \bar{u}_1 \\ -\bar{z}_2 & {}^t\overline{X}_2 & X_4 & -\bar{u}_2 \\ i\lambda & {}^t z_1 & {}^t z_2 & -\bar{a} \end{bmatrix} : \begin{array}{l} a \in \mathbf{C},\ \lambda, t \in \mathbf{R} \\ u_1, z_1 \in \mathbf{C}^p,\ u_2, z_2 \in \mathbf{C}^q \\ \text{as column vectors} \end{array} \right.$$

$$\left. X_1,\ X_4 \text{ skew Hermitian, } X_2 \text{ arbitrary complex matrix, } \operatorname{Tr} A = 0 \right\},$$

$$\mathfrak{G}_{-2} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\lambda & 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\mathfrak{G}_{-1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bar{z}_1 & 0 & 0 & 0 \\ -\bar{z}_2 & 0 & 0 & 0 \\ 0 & {}^t z_1 & {}^t z_2 & 0 \end{bmatrix} \right\}, \qquad \mathfrak{G}_0 = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & X_1 & X_2 & 0 \\ 0 & {}^t\overline{X}_2 & X_4 & 0 \\ 0 & 0 & 0 & -\bar{a} \end{bmatrix} \right\},$$

$$\mathfrak{G}_1 = \sigma \mathfrak{G}_{-1}, \qquad \mathfrak{G}_2 = \sigma \mathfrak{G}_{-2},$$

$$\mathfrak{K} = \mathfrak{s}(\mathfrak{u}(p+1) + \mathfrak{u}(q+1)),$$

$$\mathfrak{K}_0 = \left\{ \begin{bmatrix} i\mu & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 \\ 0 & 0 & X_4 & 0 \\ 0 & 0 & 0 & i\mu \end{bmatrix} \right\} = \mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(p) + \mathfrak{u}(q)),$$

$$\mathfrak{u}(1) = \left\{ \begin{bmatrix} 0 & 0 & 0 & i\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\lambda & 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\mathfrak{G}^u = \mathfrak{su}(p+q+2),$$

$$\mathfrak{su}(2)=\left\{\begin{bmatrix} ib & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{z} & 0 & 0 & -ib \end{bmatrix} : \begin{matrix} b\in\mathbf{R} \\ z\in\mathbf{C}\end{matrix}\right\},$$

$$\mathfrak{B}_0^u=\mathfrak{s}(\mathfrak{u}(1)+\mathfrak{u}(p+q)),\qquad \mathfrak{B}_0=\mathfrak{u}(p,q),$$
$$\mathfrak{su}(2)+\mathfrak{B}_0^u=\mathfrak{s}(\mathfrak{u}(2)+\mathfrak{u}(p+q)).$$

Let $M^c_{2p+1,2q+1}$ be a real hypersurface of $\mathbf{C}P^m$, $m=p+q+1$, defined by the equation

$$\sum_{i=1}^{p}|z_i|^2-\sum_{j=1}^{q}|z_{p+j}|^2-z_m\bar{z}_0-z_0\bar{z}_m=0$$

where $\{z_0,z_1,\dots,z_m\}$ is a homogeneous coordinate of $\mathbf{C}P^m$. Now $SU(Q)$ is a subgroup of $SL(m+1;\ C)$ which leaves $M^c_{2p+1,2q+1}$ invariant. Let H be the isotropy subgroup of $SU(Q)$ at the point $e=[1:0:\cdots:0]$. It is known that $M^c_{2p+1,2q+1}$ is simply connected. Therefore H is connected. Actually, $H=N_+\rtimes CU$ where

$$N_+=\left\{\begin{bmatrix} 1 & * & * \\ 0 & \delta_i^j & * \\ 0 & 0 & 1\end{bmatrix}\in H\right\},$$

$$CU=\left\{\begin{bmatrix} b & 0 & 0 \\ 0 & A_{n\times n} & 0 \\ 0 & 0 & \bar{b}^{-1}\end{bmatrix}\in H\right\}.$$

Now $K=S(U(p+1)\times U(q+1))$ while $K_0=K\cap H$ is the isotropy subgroup of K acting on $M^c_{2p+1,2q+1}$ at e and thus equal to $S(U(1)\times U(p)\times U(q))$. Moreover, $G^u=SU(p+q+2)$, $U(1)\cdot K_0=S(U(1)\times U(p)\times U(q)\times U(1))$, $U'(1)\cdot H_0^u=S(U(1)\times U(p+q)\times U(1))$ and $SU(2)\cdot H_0^u=S(U(2)\times U(p+q))$. Here the following diagram commutes.

$$M^c_{2p+1,2q+1}=\frac{S(U(p+1)\times U(q+1))}{S(U(1)\times U(p)\times U(q))}=SU(p+1,q+1)/H$$
$$\downarrow$$
$$\mathbf{C}P^p\times\mathbf{C}P^q=\frac{S(U(p+1)\times U(q+1))}{S(U(1)\times U(p)\times U(1)\times U(q))}$$
$$\to\frac{SL(p+q+2;\mathbf{C})}{H^c}=\frac{SU(p+q+2)}{S(U(1)\times U(p+q)\times U(1))}$$
$$\downarrow$$
$$\to\frac{SU(p+q+2)}{S(U(2)\times U(p+q))}$$

Note that

$$S(U(p+1)\times U(q+1))\cap S(U(1)\times U(p+q)\times U(1)) = S(U(1)\times U(p)\times U(q)).$$

Therefore our choice of H coincides with the intersection of $SU(p+1,q+1)$ and H^c.

4. Algebraic characterization and classification. Let $(\mathfrak{G}, E)$ be a gradation of the second kind for $\mathfrak{G}$ as defined in §3. Let $\mathfrak{A}^c$ be a Cartan subalgebra of $\mathfrak{G}^c$ which contains E, and let Δ denote the set of nonzero roots with respect to $\mathfrak{A}^c$. We take a basis $\{\alpha_j\}$ of Δ such that $\alpha_j(E)\geqslant 0$ for all j. Let $\tilde{\alpha}$ be the highest root with respect to this basis. If $\alpha\in\Delta$ is positive and if $\alpha(E)=2$ then $L_\alpha=\{X\in\mathfrak{G}^c\,|\,[h,X]=\alpha(h)X,\ h\in\mathfrak{A}^c\}$ is contained in $\mathfrak{G}_2^c$. Since it is clear that $\tilde{\alpha}(E)=2$, it follows that $\alpha\in\Delta$, $\alpha(E)=2$ *if and only if* $\alpha=\tilde{\alpha}$.

Let E^* be the dual of E with respect to the Killing form of $\mathfrak{G}^c$ restricted to $\mathfrak{A}^c$. Denote the Killing form of $\mathfrak{G}^c$ and the associated bilinear form on the dual space of $\mathfrak{A}^c$ by the same notation (,). We have shown just above that $\mathfrak{G}_2^c=L_{\tilde{\alpha}}$. Now by Lemmas D3 and D2 we have

$$\tilde{\alpha}=2(E,E)^{-1}E^*.$$

Moreover, we see that *any two gradations of the second kind for a given simple Lie algebra $\mathfrak{G}$ are equivalent.*

Next we give a simple algebraic characterization of our graded Lie algebra. If a gradation $(\mathfrak{G}, E)$ of the second kind for $\mathfrak{G}$ is given, let $\mathfrak{G}=\mathfrak{K}+\mathfrak{M}$ be a Cartan decomposition of $\mathfrak{G}$. Let $\mathfrak{A}_m$ be any maximal abelian subspace of $\mathfrak{M}$ containing E. The set of roots which do not vanish identically on $\mathfrak{A}_m$ will be denoted by Δ_{m}. A basis $\{\alpha_j\}$ of nonzero roots with respect to a Cartan subalgebra of $\mathfrak{G}^c$ containing $\mathfrak{A}_{\mathrm{m}}$ is taken such that $\alpha_j(E)\geqslant 0$ for all j. It is clear that the multiplicity of the highest root $\tilde{\alpha}\in\Delta_{\mathrm{m}}$ restricted to $\mathfrak{A}_{\mathrm{m}}$ must be equal to 1. In fact, if $\tilde{\alpha}\neq\alpha\in\Delta_{\mathrm{m}}$ and $\hat{\alpha}=\hat{\tilde{\alpha}}$ where $\hat{\alpha}$ denotes the restriction of α to $\mathfrak{A}_{\mathrm{m}}$, then α must be longer than $\tilde{\alpha}$ whose dual lies in $\mathfrak{A}_{\mathrm{m}}$.

Conversely, assume that there is a long root α of Δ in Δ_{m} such that the multiplicity $m_{\hat{\alpha}}$ of the restricted root $\hat{\alpha}$ is equal to 1. Let $\bar{\alpha}(h)$ be the complex conjugate of $\alpha(h)$ for $h\in\mathfrak{A}=\mathfrak{A}_{\mathfrak{k}}+\mathfrak{A}_{\mathrm{m}}$ where $\mathfrak{A}$ is the maximal abelian subalgebra of $\mathfrak{G}$ containing $\mathfrak{A}_{\mathrm{m}}$ and $\mathfrak{A}_{\mathfrak{k}}=\mathfrak{A}\cap\mathfrak{K}$. Notice that in the reverse process, we do not have E a priori. So $\mathfrak{A}_{\mathrm{m}}$ is just any maximal abelian subspace of $\mathfrak{M}$. Then $\bar{\alpha}$ is a root and $\hat{\alpha}=\hat{\bar{\alpha}}$. It follows that $\alpha=\bar{\alpha}$ by assumption and the dual of α lies in $\mathfrak{A}_{\mathrm{m}}$.

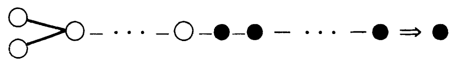
Let $a_{\beta,\alpha}$ denote the integer $2(\beta,\alpha)(\alpha,\alpha)^{-1}$ for $\alpha,\beta\in\Delta$, the set of nonzero roots with respect to the Cartan subalgebra $\mathfrak{A}^c$ of $\mathfrak{G}^c$. From Schwartz's inequality, it follows that $0\leqslant a_{\alpha,\beta}a_{\beta,\alpha}\leqslant 3$ for α, β linearly independent. If α is a long root, $|a_{\beta,\alpha}|$ must be equal to either 0 or 1 unless $\beta=\pm\alpha$. Therefore $E=2\cdot(\alpha,\alpha)^{-1}$. Since (dual of α) $\in\mathfrak{A}_{\mathrm{m}}\subset\mathfrak{G}$ will give a gradation of the second kind for $\mathfrak{G}$, we have proven the following result.

THEOREM 3. *A simple Lie algebra $\mathfrak{G}$ admits a gradation of the second kind if and only if Δ_{m} contains a root α such that α is a long root in Δ and $m_{\hat{\alpha}}=1$.*

Let Σ denote the set of restricted roots, and let (Σ, m) denote the restricted roots with multiplicities and their diagram (cf. [**L**]). Now by Lemma E1, the uniqueness of the gradation and Theorem 3, we obtain a complete list of our graded Lie algebras in Table 1.

An extended Satake diagram is constructed by adding the negative of the highest root as an extra vertex to the usual Satake diagram. For example, if $\mathfrak{so}(p, q)$ has a Satake diagram of the following type

Then the associated extended Satake diagram is



We shall next explain a method to read off $\mathfrak{B}_0$ from the extended Satake diagram of $\mathfrak{G}$. (See the Proposition below.) Consider $\mathfrak{B}_0$ acting on $\mathfrak{G}_{-1}$ through the adjoint action. Let V be a maximal $\mathfrak{B}_0$-submodule of $\mathfrak{G}_{-1}$. Then

$$\begin{aligned}\operatorname{ad}(\mathfrak{B}_0)[\sigma V, \mathfrak{G}_{-2}] &= [\mathfrak{B}_0, [\sigma V, \mathfrak{G}_{-2}]] \\ &= [\sigma V, \cancelto{0}{[\mathfrak{B}_0, \mathfrak{G}_{-2}]}] + [[\sigma V, \mathfrak{B}_0], \mathfrak{G}_{-2}] \\ &= [\sigma[V, \mathfrak{B}_0], \mathfrak{G}_{-2}] \quad \text{since } \sigma\mathfrak{B}_0 = \mathfrak{B}_0 \\ &\subseteq [\sigma V, \mathfrak{G}_{-2}].\end{aligned}$$

The above computation shows that $[\sigma V, \mathfrak{G}_{-2}]$ is an $\mathfrak{B}_0$-submodule of $\mathfrak{G}_{-1}$. By maximality of V, $[\sigma V, \mathfrak{G}_{-2}]$ is contained in V. Now it is easy to see that $\mathfrak{G}_{-2} + V + \mathfrak{G}_0 + \sigma(V) + \mathfrak{G}_2$ is an ideal of $\mathfrak{G}$. Since $\mathfrak{G}$ is simple, V must be equal to $\mathfrak{G}_{-1}$. We have shown that $\mathfrak{B}_0$ acts irreducibly on $\mathfrak{G}_{-1}$ through the adjoint action. Similarly $\mathfrak{B}_0^c$ acts also irreducibly on $\mathfrak{G}_{-1}^c$ through the adjoint representation. Therefore both $\mathfrak{B}_0$ and $\mathfrak{B}_0^c$ are reductive with $\dim_{\mathbf{C}} Z(\mathfrak{B}_0^c) \leqslant 1$ where $Z(\mathfrak{B}_0^c)$ denotes the center of $\mathfrak{B}_0^c$ (cf. [**Hu**, p. 102]). It follows that $\mathfrak{B}_0 = S(\mathfrak{B}_0) \oplus Z(\mathfrak{B}_0)$ where $S(\mathfrak{B}_0)$ denotes the semisimple part of $\mathfrak{B}_0$ and the center $Z(\mathfrak{B}_0)$ of $\mathfrak{B}_0$ has at most one dimension.

Let $\hat{\mathfrak{A}} = \mathfrak{A} \cap S(\mathfrak{B}_0)$ where $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{G}$ containing E as before. Since $E \in \mathfrak{A}$ and $Z(\mathfrak{B}_0) \subset A$, it is easy to see that $\mathfrak{A}$ is spanned by E, $\hat{\mathfrak{A}}$ and $Z(\mathfrak{B}_0)$. Let $\Delta' = \{\alpha \in \Delta \mid \alpha(E) = 0\}$ where Δ is the set of nonzero roots with respect to the Cartan subalgebra $\mathfrak{A}^c$ of $\mathfrak{G}^c$. Let $\hat{\Delta}$ be the set of nonzero roots with respect to the Cartan subalgebra $\hat{\mathfrak{A}}^c$ of $S(\mathfrak{B}_0)^c$. Set $\hat{\alpha} = \alpha \mid \hat{\mathfrak{A}}^c$ for $\alpha \in \Delta'$. Then $\hat{\alpha} \in \hat{\Delta}$. Conversely, given $\hat{\alpha} \in \hat{\Delta}$, the linear form α on $\mathfrak{A}^c$ defined by $\alpha \mid \hat{\mathfrak{A}}^c = \hat{\alpha}$, $\alpha(E) = 0$, $\alpha(Z(\mathfrak{B}_0)) = 0$ is an element of Δ'. Since $\alpha(E) = 0$ implies $\alpha(Z(\mathfrak{B}_0)) = 0$, the correspondence $\alpha \to \hat{\alpha} = \alpha \mid \hat{\mathfrak{A}}^c$ from Δ' onto $\hat{\Delta}$ is injective. We have proved that there is a linear isomorphism between Δ' and $\hat{\Delta}$.

Let $\{\alpha_j\}$ be a basis of Δ such that $\alpha_j(E) \geqslant 0$ for all j. Hence $\{\alpha_j \mid \alpha_j(E) = 0\}$ is a basis of the vector space Δ', and the corresponding set $\{\hat{\alpha}_j \mid \alpha_j(E) = 0\}$ is a basis of $\hat{\Delta}$.

TABLE 1. Simple graded Lie algebras of the second kind with dim $\mathfrak{G}_{-2} = 1$

		(Σ, m)	
AI	$\mathfrak{sl}(n,R)$	1 1 1 0—0—· · · —0	
AIII	$\mathfrak{su}(p,q)$	2 2 2 2(p-q) 0—0· · · —0⟹⓪ [1]	2 2 2 1 0—0· · · —0⟸0
BDI	$\mathfrak{so}(p,q)$	1 1 1 p-q 0—0· · · 0⟹0	1 1 1 2 0—0· · · 0⟹0
		1 1 1 1 1 0 0—0· · · 0—0< 1 0	
CI	$\mathfrak{sp}(n,R)$	1 1 1 1 0—0—· · · —0⟸0	
DIII	$\mathfrak{so}^*(2n)$	4 4 4 1 0—0· · —0⟸0	4 4 4 4 0—0—· · · —0⟹⓪[1]
EI	$\mathfrak{e}_{6(6)}$	01 1 1 | 1 1 0—0—0—0—0 1	
EII	$\mathfrak{e}_{6(2)}$	1 1 2 2 0—0⟹0—0	
EIII	$\mathfrak{e}_{6(-14)}$	6 8 [1] 0⟹⓪	
EV	$\mathfrak{e}_{7(7)}$	10 1 1 1 | 1 1 0—0—0—0—0—0 1	
EVI	$\mathfrak{e}_{7(-5)}$	4 4 1 1 0—0⟸0—0	
EVII	$\mathfrak{e}_{7(-25)}$	8 8 1 0——0⟸0	
EVIII	$\mathfrak{e}_{8(8)}$	0[1] 1 1 1 1 | 1 1 0—0—0—0—0—0—0 1	
EIX	$\mathfrak{e}_{8(-24)}$	8 8 1 1 0—0⟸0—0	
FI	$\mathfrak{f}_{4(4)}$	1 1 1 1 0—0⟸0—0	
G	$\mathfrak{G}_{2(2)}$	1 1 0⟸0	

Let S_1 be any simple ideal of $S(\mathfrak{B}_0)$. Denote the Killing forms of $\mathfrak{G}^c$ and $S(\mathfrak{B}_0)^c$ by B and $\hat{B}$ respectively. (The associated bilinear forms on duals of Cartan subalgebras are also denoted by the same symbols.) The invariant symmetric bilinear forms $B|S_1$ and $\hat{B}|S_1$ on S_1 coincide up to a positive constant. Therefore we conclude that

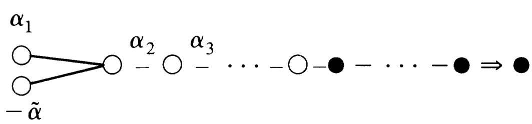
(i) $2B(\alpha_i, \alpha_j)/B(\alpha_j, \alpha_j) = 2\hat{B}(\hat{\alpha}_i, \hat{\alpha}_j)/\hat{B}(\hat{\alpha}_j, \hat{\alpha}_j)$ for α_i, α_j satisfying $\alpha_i(E) = 0$, $\alpha_j(E) = 0$, and

(iii) $B(\alpha_i, \alpha_i) > B(\alpha_j, \alpha_j)$ if and only if $\hat{B}(\hat{\alpha}_i, \hat{\alpha}_i) > \hat{B}(\hat{\alpha}_j, \hat{\alpha}_j)$ for α_i, α_j satisfying $\alpha_i(E) = 0$, $\alpha_j(E) = 0$.

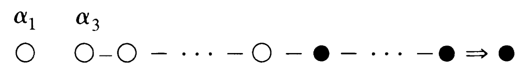
Since E^* is the scalar multiple of $\tilde{\alpha}$, the usual diagram of $S(\mathfrak{B}_0)$ is obtained by deleting the vertex $-\tilde{\alpha}$ and the vertices which are connected with $-\tilde{\alpha}$ by lines from the extended usual diagram of $\mathfrak{G}$ according to (i) and (ii). Let $\hat{\mathfrak{A}}_{\mathfrak{m}} = \mathfrak{A}_{\mathfrak{m}} \cap S(\mathfrak{B}_0)$ where $\mathfrak{A}_{\mathfrak{m}}$ is a maximal abelian subspace of $\mathfrak{M}$ containing E. (Recall that $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ is a Cartan decomposition and $\mathfrak{A}$ is always taken to be the maximal abelian subalgebra of $\mathfrak{G}$ containing $\mathfrak{A}_{\mathfrak{m}}$.) Then $\hat{\alpha}_j | \hat{\mathfrak{A}}_{\mathfrak{m}} = 0$ if and only if $\alpha_j | \mathfrak{A}_{\mathfrak{m}} = 0$ for α_j satisfying $\alpha_j(E) = 0$. Hence, the vertex $\hat{\alpha}_j$ is black if and only if the vertex α_j is black. It is also easy to see that two vertices $\hat{\alpha}_i$ and $\hat{\alpha}_j$ are connected by an arrow if and only if the vertices α_i and α_j are connected by an arrow. We have completed the proof of the following proposition.

PROPOSITION 4.4. *The diagram of the semisimple part $S(\mathfrak{B}_0)$ of $\mathfrak{B}_0$ is obtained by deleting the vertex $-\tilde{\alpha}$ and the vertices which are connected with $-\tilde{\alpha}$ by lines from the extended Satake diagram of $\mathfrak{G}$.*

Let us give a couple of examples to explain the result above. If $\mathfrak{G} = \mathfrak{so}(p, q)$, the extended Satake diagram may have the following type,

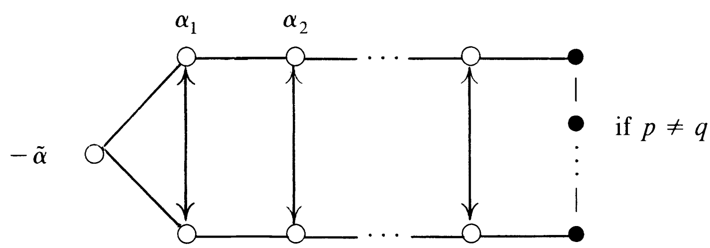


so the diagram of $S(\mathfrak{B}_0)$ is given by

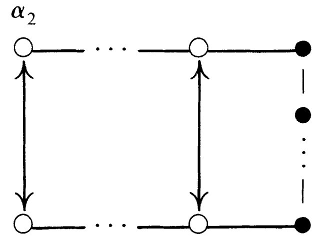


which is the diagram of $\mathfrak{sl}(2, \mathbf{R}) + \mathfrak{so}(p-2, q-2)$.

If $\mathfrak{G} = \mathfrak{su}(p, q)$, the extended Satake diagram is given by



so the diagram of $S(\mathfrak{B}_0)$ is given by



which is the diagram of $\mathfrak{su}(p-1, q-1)$.

The skew symmetric bilinear form f defined by $[X, Y] = f(X, Y)e_{-2}$ for $X, Y \in \mathfrak{G}_{-1}$ and a base $e_{-2} \in \mathfrak{G}_{-2}$ is nondegenerate by Lemma C2(4) in §1. It follows that $\mathfrak{G}_{-1}$ is even dimensional. By Lemma C3, $\mathfrak{G}_1$ is also even dimensional. Since $\dim Z(\mathfrak{B}_0) \leqslant 1$ and $\mathfrak{G}_0$ is spanned by E and $\mathfrak{B}_0$, we compute the dimension of $Z(\mathfrak{B}_0)$ by the formula

$$\dim Z(\mathfrak{B}_0) = \dim \mathfrak{G} - \dim S(\mathfrak{B}_0) - 1 \pmod 2.$$

As a result, $Z(\mathfrak{B}_0)$ is equal to $\{0\}$ for all the cases except AI and AIII.

Now we have Table 2 using the Proposition above and the argument about $Z(\mathfrak{B}_0)$.

TABLE 2

	$\mathfrak{G}$	$\mathfrak{K}_0$	$\mathfrak{B}_0$	$\mathfrak{B}_0^u$
AI	$\mathfrak{sl}(n, \mathbf{R})$	$\mathfrak{so}(n-2)$	$\mathfrak{gl}(n-2, \mathbf{R})$	$\mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(n-2))$
AIII	$\mathfrak{su}(p, q)$	$\mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(p-1) + \mathfrak{u}(q-1))$	$\mathfrak{u}(p-1, q-1)$	$\mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(p+q-2))$
BDI	$\mathfrak{so}(p, q)$	$\mathfrak{so}(2) + \mathfrak{so}(p-2) + \mathfrak{so}(q-2)$	$\mathfrak{sl}(2, \mathbf{R}) + \mathfrak{so}(p-2, q-2)$	$\mathfrak{su}(2) + \mathfrak{so}(p+q-4)$
CI	$\mathfrak{sp}(n, \mathbf{R})$	$\mathfrak{u}(n-1)$	$\mathfrak{sp}(n-1, \mathbf{R})$	$\mathfrak{sp}(n-1)$
DIII	$\mathfrak{so}^*(2n)$	$\mathfrak{su}(2) + \mathfrak{u}(n-2)$	$\mathfrak{su}(2) + \mathfrak{so}^*(2n-4)$	$\mathfrak{su}(2) + \mathfrak{so}(2n-4)$
EI	$e_{6(6)}$	$\mathfrak{so}(6)$	$\mathfrak{sl}(6, \mathbf{R})$	$\mathfrak{su}(6)$
EII	$e_{6(2)}$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3))$	$\mathfrak{su}(3, 3)$	$\mathfrak{su}(6)$
EIII	$e_{6(-14)}$	$\mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(5))$	$\mathfrak{su}(1, 5)$	$\mathfrak{su}(6)$
EV	$e_{7(7)}$	$\mathfrak{so}(6) + \mathfrak{so}(6)$	$\mathfrak{so}(6, 6)$	$\mathfrak{so}(12)$
EVI	$e_{7(-5)}$	$\mathfrak{u}(6)$	$\mathfrak{so}^*(12)$	$\mathfrak{so}(12)$
EVII	$e_{7(-25)}$	$\mathfrak{so}(2) + \mathfrak{so}(10)$	$\mathfrak{so}(2, 10)$	$\mathfrak{so}(12)$
EVIII	$e_{8(8)}$	$\mathfrak{su}(8)$	$e_{7(7)}$	$e_{7(-133)}$
EIX	$e_{8(-24)}$	$e_6 + \mathfrak{so}(2)$	$e_{7(-25)}$	$e_{7(-133)}$
FI	$f_{4(4)}$	$\mathfrak{u}(3)$	$\mathfrak{sp}(3, \mathbf{R})$	$\mathfrak{sp}(3)$
G	$\mathfrak{G}_{2(2)}$	$\mathfrak{so}(2)$	$\mathfrak{sl}(2, \mathbf{R})$	$\mathfrak{su}(2)$

Note that $\mathfrak{K}_0$ is a maximal compactly imbedded subalgebra of $\mathfrak{B}_0$, and that $\mathfrak{B}_0^u$ is the compact dual of $\mathfrak{B}_0$.

From Table 1, we see that there are five classes of classical Lie algebras of our type. We have dealt with AIII and CI in §3 in detail. Similarly we can work out the other three cases. Here we only give E's and σ's and list the essential spaces appearing in Theorem 1 in §2 for reference.

AI. $$\mathfrak{G} = \mathfrak{sl}(n+2, \mathbf{R}), \quad \sigma X = -{}^tX, \qquad E = \begin{bmatrix} 1 & & \\ & 0_{n\times n} & \\ & & -1 \end{bmatrix},$$

BDI. $$\mathfrak{G} = \mathfrak{so}(p,q), \quad \sigma X = -{}^tX, \qquad E = \left[\begin{array}{ccc|ccc} & & & 1 & 0 & \\ & 0 & & 0 & 1 & \\ & & & & & 0 \\ \hline 1 & 0 & & & & \\ 0 & 1 & & & 0 & \\ & & 0 & & & \end{array}\right],$$

DIII. $$\mathfrak{G} = \mathfrak{so}(2n), \quad \sigma X = -{}^t\overline{X} = \overline{X}, \qquad E = \left[\begin{array}{ccc|ccc} 0 & i & & & & \\ -i & 0 & & & 0 & \\ & & 0 & & & \\ \hline & & & 0 & -i & \\ & 0 & & i & 0 & \\ & & & & & 0 \end{array}\right].$$

TABLE 3

	K/K_0	$K/U(1)\cdot K_0$	$G^u/SU(2)\cdot H_0^u$
AI	$\dfrac{SO(m)}{S(\{\pm1\}\times O(m-2)\times\{\pm1\})}$	$\dfrac{SO(m)}{S(O(2)\times O(m-2))}$	$\dfrac{SU(m)}{S(U(2)\times U(m-2))}$
AIII	$M^c_{2p+1,2q+1}$	$\mathbf{C}P^p\times\mathbf{C}P^q$	$\dfrac{SU(p+q+2)}{S(U(2)\times U(p+q))}$
BDI	$\dfrac{SO(p)\times SO(q)}{SO(2)\times SO(p-2)\times SO(q-2)}$	$Q_{p-2}(\mathbf{C})\times Q_{q-2}(\mathbf{C})$	$\dfrac{SO(n)}{SO(4)\times SO(n-4)}$
CI	$\mathbf{R}P^{2n-1}=\dfrac{U(n)}{\{\pm1\}\times U(n-1)}$	$\mathbf{C}P^{n-1}=\dfrac{U(n)}{U(1)\times U(n-1)}$	$\mathbf{H}P^{n-1}=\dfrac{Sp(n)}{Sp(1)\times Sp(n-1)}$
DIII	$\dfrac{U(n)}{SU(2)\times U(n-2)}$	$\dfrac{U(n)}{U(2)\times U(n-2)}$	$\dfrac{SO(2n)}{SO(4)\times SO(2n-4)}$

In BDI, $Q_{p-2}(\mathbf{C})$ denotes a complex quadric of complex $p-2$ dimension and $n = p+q$.

Appendix. A quaternion algebra on a real vector space V is an algebra of linear transformations of V which is isomorphic to the algebra of real quaternions, and whose unit element is the identity transformation of V.

Let M be a Riemannian manifold. The linear holonomy (group) I_x at $x \in M$ consists of all linear transformations of the tangent space M_x obtained from parallel translation along curves from x to x. A set A_x of linear transformations of M_x is called I_x-invariant if $gA_xg^{-1} = A_x$ for every $g \in I_x$. A set A of fields of linear transformations of all tangent spaces of M is called parallel if, given $x, y \in M$ and a curve σ from x to y, the parallel translation τ_σ along σ satisfies $\tau_\sigma A_x \tau_\sigma^{-1} = A_y$. We say that A_x is I_x-invariant if and only if A_x extends to a parallel set of fields of linear transformations of tangent spaces. The extension is unique in that case.

A *quaternionic structure* on a Riemannian manifold M is a parallel field A of quaternion algebras A_x on the tangent spaces M_x, such that every unimodular element of A_x is an orthogonal linear transformation of M_x. Let M be a Riemannian manifold with a quaternionic structure A. The holonomy I_x at $x \in M$ is said to have quaternion scalar part if $I_x = SU(2) \cdot \Phi_x$ where Φ_x is the centralizer of A_x and $SU(2) = Sp(1)$ is the group of unimodular A_x-scalar transformations. For our compact simply connected symmetric space $M = G^u/SU(2) \cdot H_0^u$ in §2, $\mathrm{Ad}(\exp(\pi\alpha/2))$ and $\mathrm{Ad}(\exp(\pi i E/2))$ form an anticommuting pair of transformations of square -1 on $M_0 \cong (1+\sigma)\mathfrak{G}_{-1} + i(1-\sigma)\mathfrak{G}_{-1}$. Therefore $\mathrm{Ad}(SU(2))|_{M_0}$ generates a quaternion algebra on M_0. As $\mathrm{Ad}(SU(2) \cdot H_0^u)|_{M_0}$ is the linear holonomy containing $\mathrm{Ad}(SU(2))$ as a normal subgroup, $G^u/SU(2) \cdot H_0^u$ admits a quaternionic structure in which the holonomy has quaternion scalar part.

Let M be a compact simply connected quaternionic symmetric space. Let A denote its quaternionic structure. An involutive isometry σ with a fixed point $0 \in M$ is called antiquaternionic if $A_0 = d\sigma_0 A_0 d\sigma_0^{-1}$ and $\{q \in A_0 \mid q d\sigma_0 = d\sigma_0 q\}$ is a complex subfield of A_0. It follows that $\mathrm{Ad}\,\sigma$ leaves $SU(2)$ and $F(\mathrm{Ad}\,\sigma, SU(2)) \cong U(1)$ invariant. Since $SU(2) \cdot H_0^u$ contains the Cartan involution, $\operatorname{rank} G^u = \operatorname{rank}(SU(2) \cdot H_0^u)$. In other words, $SU(2) \cdot H_0^u$ contains a maximal torus T of G^u. In particular, $G^u = I_0 M$ is centerless. For simplicity of G^u and more details, see [**W**].

Suppose now that in the definition of equivalence relation of the pairs (M_1, σ_1) and (M_2, σ_2) in §2, we remove the condition that f carries A_1 to A_2. Then an easy argument shows that $\mathfrak{G}_1$ is isomorphic to $\mathfrak{G}_2$. By the uniqueness of the gradation, $(\mathfrak{G}_1, E_1)$ is equivalent to $(\mathfrak{G}_2, E_2)$. Therefore (M_1, σ_1) is equivalent to (M_2, σ_2) in the original sense.

It is known that we have eight families of compact simply connected quaternionic symmetric spaces, namely $SU(n+2)/S(U(2) \times U(n))$, $SO(n+4)/SO(4) \times SO(n)$, and $Sp(n+1)/Sp(1) \times Sp(n)$ for classical cases, and $E_6/Sp(1) \cdot SU(6)$, $E_7/Sp(1) \cdot \mathrm{Spin}(12)$, $E_8/Sp(1) \cdot E_7$, and $F_4/Sp(1) \cdot Sp(3)$, $G_2/SO(4)$ for exceptional cases.

Added in proof. The author would like to express his gratitude to both the referee and the editor for their help in improving the English expression of the original manuscript.

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